A Generalized Dimension Function for K_0 of C^*-Algebras

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This exposition has been guided by John Roe's lecture notes on K-theory. In what follows we develop a generalized dimension function that can assist with analyzing the structure and classification of C^*-algebras.

Note that if τ is a trace on an algebra A then,

$$\tau_n \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \sum_{j=1}^{n} \tau(a_{jj})$$

is a trace on $M_n(A)$. If we need to work with matrices of different sizes we *stabilize* the matrices by adjoining rows and columns of zeros. Thus, we write τ_∞ and $M_\infty(A)$, where $M_\infty(A)$ is the direct limit of the connecting maps

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

and τ_∞ is the trace on $M_\infty(A)$ by the universal property of a direct limit. We denote the unitalization of A by $\widetilde{A}$. Note that if τ is a trace on A then $\widetilde{\tau} : \widetilde{A} \to \mathbb{F}$ defined by $(a, \lambda) \mapsto \tau(a) + \lambda$ is a trace on $\widetilde{A}$. Indeed, observe that $\tau\Big((a, \lambda)(b, \mu)\Big) = \tau\Big((ab + \lambda b + \mu a, \lambda\mu)\Big) = \tau(ab) + \lambda\tau(b) + \mu\tau(a) + \lambda\mu = \tau(ba) + \mu\tau(a) + \lambda\tau(b) + \mu\lambda = \tau\Big((b, \mu)(a, \lambda)\Big)$.

Definition 0.1 (The monoid $V(A)$)**.** We call two projections $p, q \in M_\infty(A)$ equivalent if they are Murray-von Neumann equivalent. That is $p \sim q$ if there exists a $v \in M_\infty(A)$ such that $p = v^*v$ and $q = vv^*$. Then $V(A)$ is the set of all such equivalence classes of projections where a element in $V(A)$ is denoted by $[\cdot]$. We make $V(A)$ a moniod by defining addition as $[p] + [q] = [p \oplus q]$

Note that, by Wegge-Olsen 5.2.10 and 5.2.12: Murray-von Neumann, unitary, and homotopy equivalence all define the same equivalence classes in $M_\infty(A)$. We denote by $K_{00}(A)$ the Grothendieck group which turns the moniod $V(A)$ into a group by considering the formal differences $[p] - [q]$ of the equivalence classes in $V(A)$ in a similar way as the integers are constructed from the natural numbers. See Wegge-Olsen appendix G for a complete construction. The universal property of Grothendieck groups lets us extend a homomorphism $\alpha_* : V(A) \to V(B)$ induced by some morphism $\alpha : A \to B$ to a group homomorphism $\alpha_* : K_{00}(A) \to K_{00}(B)$. The group $K_0(A)$ is defined using this functoriality by

$$K_0(A) := \mathrm{Ker}\Big(\pi_* : K_{00}(\widetilde{A}) \to \mathbb{Z}\Big).$$

Theorem 0.2. *Let $\mathbb{F}$ be a field, and let add$(\mathbb{F})$ be the elements of $\mathbb{F}$ viewed as an additive group. If A is an algebra over the field $\mathbb{F}$ and τ is a trace on A. Then the map $\dim_\tau : K_0(A) \to \mathrm{add}(\mathbb{F})$ defined by $\dim_\tau([p]) = \tau_\infty(p)$ is a group homomorphism.*

Proof. Note that $[p] = [q]$ in $K_0(\widetilde{A})$ if and only if there is a idempotent r and x, y, with $xy = p \oplus r$ and $yx = q \oplus r$. Thus, $\widetilde{\tau}_\infty(p) + \widetilde{\tau}_\infty(r) = \widetilde{\tau}_\infty(p \oplus r) = \widetilde{\tau}_\infty(xy) = \widetilde{\tau}_\infty(yx) = \widetilde{\tau}_\infty(q \oplus r) = \widetilde{\tau}_\infty(q) + \widetilde{\tau}_\infty(r)$. Hence, $\widetilde{\tau}_\infty(p) = \widetilde{\tau}_\infty(q)$ and so $\dim_{\widetilde{\tau}}$ is well defined. Thus, we can define $\dim_\tau$ as the restriction of $\dim_{\widetilde{\tau}}$ to $K_0(A)$. That it is a homomorphism follows from linearity. □

However, what if we do not have a trace that is defined for every element of A? For instance $A = \mathcal{K}(\ell^2\mathbb{N})$. For this we develop a method to use an unbounded trace; that is a map τ that is defined on a dense subset of A for which τ is a trace.

1 Unbounded Traces

Definition 1.1 (Tracial Weight)**.** By A^+ we shall mean the positive elements of our C^*-algebra A. That is elements of the form a^*a. A *tracial weight* on a C*-algebra A is a function $\tau : A^+ \to [0, \infty]$ such that $\tau(\lambda_1 a_1 + \lambda_2 a_2) =$

$\lambda_1\tau(a_1)+\lambda_2\tau(a_2)$ for all $\lambda_1,\lambda_2\geq 0$ and all $a_1,a_2\in A^+$; and such that $\tau(a^*a)=\tau(aa^*)$ for all $a\in A$.

Definition 1.2 (Lower Semicontinuous)**.** A tracial weight is called *lower semicontinuous* if $\tau(\lim_{n\to\infty} a_n)\leq \liminf_{n\to\infty}\tau(a_n)$ for all norm convergent sequences $\{a_n\}$ in A^+.

Definition 1.3 (Densely Defined)**.** A tracial weight is *densely defined* if the set $\{a\in A^+:\tau(a)<\infty\}$ is dense in A^+.

Definition 1.4 (Unbounded Traces)**.** A tracial weight that is lower semicontinuous and is densely defined is called an *unbounded trace*.

Lemma 1.5. *If I is a dense $*$-ideal (not necessarily closed) of a C^*-algebra A then $\Lambda_{I^+}=\{a\in I^+:\|a\|\leq 1\}$ is an upwards directed set. Moreover, if $u_\alpha=\alpha\in\Lambda_{I^+}$ then $(u_\alpha)_{\alpha\in\Lambda_{I^+}}$ is an approximate unit for A.*

Proof. (This proof is an adaption of Murphy's theorem 3.1.1) First, if $a\in A^+$ then $(1+a)\in \mathrm{Inv}(A)$ by the functional calculus. So if $a\leq b \implies 1+a\leq 1+b$, then by Murphy (2.2.5) $(1+a)^{-1}\geq(1+b)^{-1}$ and so $1-(1+a)^{-1}\leq 1-(1+b)^{-1}$ Note that, $a(1+a)^{-1}=1-(1+a)^{-1}$, and so

$$a\leq b \implies a(1+a)^{-1}\leq b(1+b)^{-1},\quad \text{for all } a,b\in A^+ \tag{1}$$

Observe that, by the functional calculus, $a(1+a)^{-1}$ is positive and $\|a(1+a)^{-1}\|\leq 1$ whenever $a\in A^+$. Moreover, if $a\in I^+$ since I is an ideal, $a(1+a)^{-1}\in I\cap A^+=I^+$. Thus, $a(1+a)^{-1}\in\Lambda_{I^+}$ whenever $a\in I^+$. Next, let $a,b\in\Lambda_{I^+}$, and let $a'=a(1-a)^{-1},\ b'=b(1-b)^{-1}$ which both exist and are positive since $\|a\|,\|b\|<1$. Additionally, $a',b'\in I\cap A^+=I^+$ since I is an ideal. Note that since $a',(1+a')^{-1}\in C^*(a,1)$ which is commutative, $a'=a(1-a)^{-1} \iff a'(1-a)=a \iff a'=a+a'a \iff a'=(1+a')a \iff (1+a')^{-1}a'=a \iff a=a'(1+a')^{-1}$. Set $c=(a'+b')(1+a'+b')^{-1}$ and note that, since I^+ is closed under addition, $c\in\Lambda_{I^+}$. Then since $a'\leq a'+b'$, by (1) we have that $a=a'(1+a')^{-1}\leq(a'+b')(1+a'+b')^{-1}=c$ and similarly $b\leq c$. Thus, Λ_{I^+} is an upwards directed set. Moreover, if we set $u_\alpha=\alpha\in\Lambda_{I^+}$, then $(u_\alpha)_{\alpha\in\Lambda_{I^+}}$ is an upwards directed net of positive elements.

Next, since Λ_{A^+} linearly spans A it suffices to show that $a=\lim_\alpha u_\alpha a=\lim_\alpha a u_\alpha$ for all $a\in\Lambda_{A^+}$. Let $\epsilon>0$ be given. Define Ω_a to be the character space of the C^*-algebra generated by a and for $\omega\in\Omega_a$ define $\hat{a}(\omega)=\omega(a)$. By the Gelfand representation, $\varphi:C^*(a)\to C_0(\Omega_a)$, the set $K=\{\omega\in\Omega_a:$

$|\hat{a}(\omega)| \geq \frac{\epsilon^2}{2}\}$ is compact. Thus, by Urysohn's lemma there exists a continuos function $g : \Omega_a \to [0,1]$ of compact support such that $g(\omega) = 1$ for all $\omega \in K$. Then $\left\|a - \varphi^{-1}(g)a\right\| = \|\hat{a} - g\hat{a}\|_\infty < \frac{\epsilon^2}{2}$. Note that $\varphi^{-1}(g) \in \Lambda_{A^+}$. Thus, since Λ_{I^+} is dense in Λ_{A^+}, there exists an $\alpha_0 \in \Lambda_{I^+}$ such that $\left\|\alpha_0 - \varphi^{-1}(g)\right\| < \frac{\epsilon^2}{2}$. Observe that:

$$\|a - \alpha_0 a\| = \left\|a - \varphi^{-1}(g)a + \varphi^{-1}(g)a - \alpha_0 a\right\|$$

$$\leq \left\|a - \varphi^{-1}(g)a\right\| + \left\|\varphi^{-1}(g)a - \alpha_0 a\right\| \leq \frac{\epsilon^2}{2} + \|a\| \left\|\varphi^{-1}(g) - \alpha_0\right\| < \epsilon^2.$$

Then for $\alpha \in \Lambda_{I^+}$ such that $\alpha \geq \alpha_0$, we have $1 - u_\alpha \leq 1 - u_{\alpha_0}$ and so by Murphy (2.2.5), $a(1-u_\alpha)a \leq a(1-u_{\alpha_0})a$. Thus,

$$\|a - u_\alpha a\|^2 = \|(1-u_\alpha)a\|^2 = \left\|(1-u_\alpha)^{\frac{1}{2}}(1-u_\alpha)^{\frac{1}{2}}a\right\|^2$$

$$\leq \left\|(1-u_\alpha)^{\frac{1}{2}}\right\|^2 \left\|(1-u_\alpha)^{\frac{1}{2}}a\right\|^2 \leq \left\|(1-u_\alpha)^{\frac{1}{2}}a\right\|^2$$

$$= \left\|a^*(1-u_\alpha)^{\frac{1}{2}}(1-u_\alpha)^{\frac{1}{2}}a\right\| = \|a(1-u_\alpha)a\| \leq \|a(1-u_{\alpha_0})a\|$$

$$\leq \|a\| \, \|(1-u_{\alpha_0})a\| < \epsilon^2$$

Hence, $\|a - u_\alpha a\| < \epsilon$ and a similar argument shows that $\|a - au_\alpha\| < \epsilon$ whenever $\alpha \geq \alpha_0$. Thus, $a = \lim_\alpha u_\alpha a = \lim_\alpha a u_\alpha$. □

Lemma 1.6. *Let τ be an unbounded trace on a C^*-algebra A. Let $I^+ = \{a \in A^+ : \tau(a) < \infty\}$ and let I be the linear span of I^+. The following claims will show that I is dense in A and admits a norm $|||\cdot|||$ under which I becomes a Banach algebra. Moreover, $|||\cdot|||$ satisfies the inequality*

$$|||xy||| \leq \|x\| \, |||y||| + |||x||| \, \|y\| \, .$$

Claim 1.6.1. *I is a dense $*$-ideal in A and τ extends uniquely to a linear functional on I which is real on self adjoint elements.*

Proof. Recall that every $a \in A$ can be written as $a = b + ic$ where $b, c \in A_{sa}$. Next, let $c \in A_{sa}$ be arbitrary. Then $c^2 = c^*c$ is positive and so $(c^2)^{1/2} = |c| \in A^+$. Let $c^+ = \frac{1}{2}(|c| + c)$, and $c^- = \frac{1}{2}(|c| - c)$. Observe that for $\gamma \in \Omega_c$, $\widehat{c^+}(\gamma) = \frac{1}{2}(|\gamma(c)| + \gamma(c))$ and since $c \in A_{sa}$, $\gamma(c) \in \mathbb{R}$ so $\widehat{c^+} = 0$ or $|\gamma(c)|$. Hence, by the Gelfand representation, $\sigma(c^+) = \widehat{c^+}(\Omega_c) \subset [0, \infty)$. A similar argument shows that $c^- \in A^+$ also. Thus, since $c = c^+ - c^-$, every self adjoint element of A is the real linear combination of two positive elements. This shows that $\text{span}(A^+) = A$ and since I^+ is dense in A^+ it follows that I is dense in A.

Next, by separating $a \in I$ into its real and imaginary parts to show that linearly extending τ to I is well defined it suffices to show that linearly extending τ to real linear combinations of elements of I^+ is well defined. Suppose that

$$\sum_{k=1}^{m} x_k a_k = \sum_{j=1}^{n} y_j a_j$$

where the x_k's, y_j's $\in \mathbb{R}$ and the a_k's , a_j's $\in I^+$. Observe that,

$$\sum_{x_k \in \mathbb{R}^+} x_k a_k + \sum_{x_k \in \mathbb{R}^-} x_k a_k = \sum_{k=1}^{m} x_k a_k = \sum_{j=1}^{n} y_j a_j = \sum_{y_j \in \mathbb{R}^+} y_j a_j + \sum_{y_j \in \mathbb{R}^-} y_j a_j$$

$$\iff \sum_{x_k \in \mathbb{R}^+} x_k a_k - \sum_{y_j \in \mathbb{R}^-} y_j a_j = \sum_{y_j \in \mathbb{R}^+} y_j a_j - \sum_{x_k \in \mathbb{R}^-} x_k a_k.$$

Then by definition 1.1 we have

$$\sum_{x_k \in \mathbb{R}^+} x_k \tau(a_k) - \sum_{y_j \in \mathbb{R}^-} y_j \tau(a_j) = \sum_{y_j \in \mathbb{R}^+} y_j \tau(a_j) - \sum_{x_k \in \mathbb{R}^-} x_k \tau(a_k)$$

$$\iff \sum_{x_k \in \mathbb{R}^+} x_k \tau(a_k) + \sum_{x_k \in \mathbb{R}^-} x_k \tau(a_k) = \sum_{y_j \in \mathbb{R}^+} y_j \tau(a_j) + \sum_{y_j \in \mathbb{R}^-} y_j \tau(a_j)$$

$$\text{and so } \sum_{k=1}^{m} x_k \tau(a_k) = \sum_{j=1}^{n} y_j \tau(a_j)$$

Thus, linearly extending τ to I is well defined. Moreover, writing $c \in I_{sa}$ as $c = c^+ - c^-$ we have $\tau(c) = \tau(c^+) - \tau(c^-) \in \mathbb{R}$.

To show that I is an ideal we proceed indirectly. First, let

$$J = \{x \in A : \tau(x^* x) < \infty\}.$$

Note that if $a \leq b$, i.e. $b - a \in A^+$ then $\tau(b) - \tau(a) \in [0, \infty]$ since τ is positive.

Hence, $\tau(a) \le \tau(b)$. Thus, for $x, y \in J$, the inequality

$$(x+y)^*(x+y) \le (x+y)^*(x+y) + (x-y)^*(x-y) = 2(x^*x + y^*y)$$

Shows that J is closed under addition. Moreover, by the proof of 2.2.5(c) (Murphy), $\|a\|^2 = \|a^*a\| \ge a^*a$ so by 2.2.5(b)(Murphy) $x^*a^*ax \le \|a\|^2 x^*x$. Thus, if $x \in J$ and $a \in A$, then $ax \in J$ and so J is a left ideal. Additionally, since $\tau(x^*x) = \tau(xx^*)$, $x \in J$ implies that $x^* \in J$ and so J is $*$-closed. Then since J is a left ideal, for $x \in J$, $a^*x^* \in J \implies (a^*x^*)^* = xa \in J$ so J is a two-sided ideal. Next consider $J^2 = \text{span}\{xy : x, y \in J\}$. Clearly J^2 is closed under addition; and since J is a two sided ideal that is $*$-closed, so is J^2. Then for $x, y \in J$, let $z_n = i^n x + y$ so that $z_n \in J$. The polarization identity,

$$x^*y = \frac{1}{4}\sum_{n=0}^{3} i^n z_n^* z_n,$$

shows that J^2 is the linear combination of elements of I^+. Hence, $J^2 \subseteq I$. Next, if $a \in I^+$, then $a^{1/2}$ is positive so that $a^{1/2} \in J$ and so $a \in J^2$. Since such elements generate I we have that $I \subseteq J^2$ and so $I = J^2$. Hence, I is a two sided $*$-ideal ☐

Claim 1.6.2. *The ideal I is hereditary; i.e. if $0 \le a \le a'$, then $a' \in I$ implies that $a \in I$.*

Proof. Since $a \le a'$, $a' - a \in A^+$ so, $\tau(a' - a) = \tau(a') - \tau(a) \ge 0$
$\implies \infty \ge \tau(a') \ge \tau(a) \ge 0$. ☐

Claim 1.6.3 (Roe)**.** *The (extended) linear functional τ has the property that $\tau(xy) = \tau(yx)$ whenever $x \in I$ and $y \in A$. In particular, it is a trace (in the algebraic sense) on I.*

Proof. Since $x \in I$, $x = \sum_{k=1}^{n} \lambda_k x_k$ where $x_k \in I^+$. Writing $y = a + ib$ where $a, b \in A_{sa}$, we have that; $yx = \sum_{k=1}^{n}(\lambda_k a x_k + i\lambda_k b x_k)$, and $xy = \sum_{k=1}^{n}(\lambda_k x_k a + i\lambda_k x_k b)$. Since τ is linear without loss of generality we may assume that $x \in I^+$ and $a \in A_{sa}$. First consider the case where $x, y \in J$. Then the polarization identity gives us that,

$$\tau(xy) = \tau(x^*y) = \tau(yx^*) = \tau(yx)$$

since $\tau(z_n^* z_n) = \tau(z_n z_n^*)$ by definition. For the general case, since $x \in I^+$ it follows that $x^{1/2} \in J$. Since J is a two sided ideal $yx^{1/2} \in J$ and $x^{1/2}y \in J$. Thus, using the previous result twice we have that

$$\tau(xy) = \tau(x^{1/2}x^{1/2}y) = \tau(x^{1/2}yx^{1/2}) = \tau(yx^{1/2}x^{1/2}) = \tau(xy).$$

□

Claim 1.6.4. *For all $x, y \in J$, $\langle \cdot, \cdot \rangle_{HS\tau}$ defined by $\langle x, y \rangle_{HS\tau} = \tau(x^*y)$ is a semi-inner product. Moreover, $\|\cdot\|_{HS\tau} = |\langle \cdot, \cdot \rangle_{HS\tau}|^{1/2}$ has the property that $\|ax\|_{HS\tau} \leq \|a\| \|x\|_{HS\tau}$ for all $x \in J$ and $a \in A$.*

Proof. First we show that $\langle \cdot, \cdot \rangle_{HS\tau}$ is a semi-inner product. Let $\alpha, \beta \in \mathbb{C}$ and let $x, y, z \in J$. Observe that:

1. $\tau\big((\alpha x + \beta y)^* z\big) = \tau\big((\overline{\alpha}x^* + \overline{\beta}y^*)z\big) = \overline{\alpha}\tau(x^*z) + \overline{\beta}\tau(y^*z)$
2. $\tau\big(x^*(\alpha y + \beta z)\big) = \alpha\tau(x^*y) + \beta\tau(x^*z)$
3. $\tau(x^*x) \geq 0$ by the definition of τ.
4. Let z_n be defined as before. Since $z_n^* z_n \in I^+$ it follows that $\tau(z_n^* z_n) = \overline{\tau(z_n^* z_n)}$. Expanding both for $n = 1, 2$, some algebra shows that , $\overline{\tau(x^*y)} = \tau(y^*x)$.

Lastly,
$x^*a^*ax \leq \|a\|^2 x^*x \implies \tau\big((ax)^*ax\big) \leq \|a\|^2 \tau(x^*x) \implies \|ax\|_{HS\tau} \leq \|a\| \|x\|_{HS\tau}$.
□

Claim 1.6.5. *If $a \in I$ then $|a| = (a^*a)^{\frac{1}{2}} \in I$. Moreover, for all $a \in I$ and $b \in A$ we have that $|\tau(ab)| \leq \|b\| \tau(|a|)$.*

Proof. We use semicontinuity and an approximation argument. First consider the case where $a \in I^+$. We have that,

$$|\tau(ab)| = |\langle a^{1/2}, a^{1/2}b \rangle| \leq \left\| a^{1/2} \right\|_{HS\tau} \left\| a^{1/2}b \right\|_{HS\tau} \leq \left\| a^{1/2} \right\|_{HS\tau}^2 \|b\| = \tau(a) \|b\| \tag{2}$$

Next, let $a \in I$. since $a = \sum_{k=1}^{n} \lambda_k a_k$ where $a_k \in I^+$ for all k. We have that,

$$|\tau(ab)| = \left| \sum_{k=1}^{n} \lambda_k \tau(a_k b) \right| \leq \sum_{k=1}^{n} |\lambda_k| \tau(a_k) \|b\| = C \|b\| \tag{3}$$

where the last inequality is given by (2) and $C > 0$ is a constant depending on a. Now, for $a \in I$ let $w_n = a(1/n + a^*a)^{1/2}$. Since $a \in I$ and I is an ideal, $w_n \in I$. Define $h_n : \sigma(a^*a) \to \mathbb{R}$ by $h_n(x) = (1/n + x)^{-1/2}x$ where $\sigma(a)$ is the spectrum of a. Then $w_n^*a = h_n(a^*a)$, so $\sigma(w_n^*a) = h_n(\sigma(a^*a)) \subseteq [0, \infty)$. Thus, $\{w_n^*a\}$ is a sequence of positive elements. Now consider, $g_n : \sigma(a^*a) \to \mathbb{R}$ defined by $g_n(x) = (1/n+x)^{-1/2}x - x^{1/2}$. Observe that $\lim_{n\to\infty} g_n = 0$, so there exists an N such that $|g_n| < \epsilon$ whenever $n \geq N$. Hence, $\sigma(g_n(a^*a)) = g_n(\sigma(a^*a)) \subseteq (-\epsilon, \epsilon)$ whenever $n \geq N$ and since $g_n(a^*a)$ is self-adjoint, w_n^*a converges in norm to $|a|$. By inequality (3) the sequence $\tau(w_n^*a)$ is bounded, so by lower semicontinuity, $\tau(|a|) < \infty$. Thus, $|a| \in I^+ \subset I$

To show that $|\tau(ab)| \leq \|b\| \, \tau(|a|)$ for all $a \in I,\ b \in A$ we first note that $\|w_n\| \leq 1$ for all n. Indeed, consider the continuous functions $f_n : \sigma(a^*a) \to \mathbb{R}$ defined by $f_n(x) = (\frac{1}{n} + x)^{-\frac{1}{2}} x (\frac{1}{n} + x)^{-\frac{1}{2}}$. Then for all n, $0 \leq f_n < 1$, and so $\sigma(f_n(a^*a)) = f_n(\sigma(a^*a)) \subseteq [0, 1)$. Moreover, since $f_n(a^*a) = w_n^*w_n \in A_{sa}$, $\|w_n^*w_n\| \in [0, 1]$ and so, $\|w_n\| \leq 1$ for all n. Next, let $\epsilon > 0$ be given. Let $a \in I$ and $b \in A$ be fixed but arbitrary. Then, since I is dense in A, we may choose $b' \in I$ such that

$$\|b' - b\| < \min\left\{\frac{\epsilon}{2\tau(|a|)}, \frac{\epsilon}{2C}\right\}. \tag{4}$$

Notice that, by (3), the function $\tau_{b'} : A \to \mathbb{C}$ defined by $c \mapsto \tau(cb')$ is a bounded linear functional on A and so is continuos. Thus, for any sequence $\{c_n\}$ in A norm converging to c we have that

$$\tau(cb') = \lim_{n\to\infty} \tau(c_n b') \tag{5}$$

Observe that, $\|w_n|a| - a\|^2 = \|(w_n|a| - a)^*(w_n|a| - a)\| \to 0$ by using the functional calculus as we did with g_n. Thus, using (5) we have,

$$|\tau(ab')| = \lim_{n\to\infty} \left|\tau(w_n|a|b')\right| \overset{1.6.3}{=} \lim_{n\to\infty} \left|\tau(|a|b'w_n)\right| \overset{(2)}{\leq} \tau(|a|)\, \|b'\| \sup_n \|w_n\| \leq \tau(|a|)\, \|b'\|.$$

Observe that,

$$\Big||\tau(ab)| - |\tau(ab')|\Big| \leq \left|\tau\big(a(b - b')\big)\right| \overset{(3)}{\leq} C\, \|b - b'\| \overset{(4)}{<} \frac{\epsilon}{2}$$

so that, $|\tau(ab)| - \frac{\epsilon}{2} < |\tau(ab')|$. Also,

$$\big|\tau(|a|)\,\|b'\| - \tau(|a|)\,\|b\|\,\big| \leq \tau(|a|)\,\|b' - b\| < \frac{\epsilon}{2}$$

so that, $\tau(|a|)\,\|b'\| < \tau(|a|)\,\|b\| + \frac{\epsilon}{2}$. Hence, $|\tau(ab')| \leq \tau(|a|)\,\|b'\|$ implies that $|\tau(ab)| < \tau(|a|)\,\|b\| + \epsilon$. Then, since ϵ was arbitrary we obtain the desired inequality. □

Claim 1.6.6. *The inequality $\tau(|a+b|) \leq \tau(|a|) + \tau(|b|)$ holds for all $a, b \in I$.*

Proof. Note that for all x such that $\|x\| \leq 1$, $|\tau(ax)| \leq \tau(|a|)\,\|x\| \leq \tau(|a|)$ by 1.6.5. Thus, $\sup_{\|x\|\leq 1} |\tau(ax)| \leq \tau(|a|)$. Next, since $\|w_n^*\| = \|w_n\| \leq 1$ lower semicontinuity gives us that $\tau(|a|) \leq \liminf_{n\to\infty} \tau(w_n^* a) \leq \sup_{\|x\|\leq 1} |\tau(ax)|$. Hence,

$$\tau(|a|) = \sup_{\|x\|\leq 1} |\tau(ax)| \tag{6}$$

Thus,

$$\tau(|a+b|) = \sup_{\|x\|\leq 1} |\tau(ax) + \tau(bx)| \leq \sup_{\|x\|\leq 1} |\tau(ax)| + \sup_{\|x\|\leq 1} |\tau(bx)| = \tau(|a|) + \tau(|b|)$$

□

Remark. Note that 1.6.6 implies that,

$$\tau(|ab|) = \sup_{\|x\|\leq 1} |\tau(abx)| \overset{1.6.5}{\leq} \sup_{\|x\|\leq 1} \tau(|a|)\,\|b\|\,\|x\| = \tau(|a|)\,\|b\| \tag{7}$$

Remark. For any $a \in A$ using the functional calculus on $C^*(a^*a, 1)$, it follows that, $(a^*a)^{\frac{1}{2}} - (\frac{1}{n} + a^*a)^{-\frac{1}{2}} a^*a \in A^+$ for all n so that $\tau(w_n^* a) \leq \tau(|a|)$. Thus, $\limsup_{n\to\infty} \tau(w_n^* a) \leq \tau(|a|) \leq \liminf_{n\to\infty} \tau(w_n^* a)$ by lower semicontinuity. Hence,

$$\lim_{n\to\infty} \tau(w_n^* a) = \tau(|a|) \tag{8}$$

for all $a \in A$.

Claim 1.6.7. *If τ is an unbounded trace; the inequality $\tau(|a+b|) \leq \tau(|a|) + \tau(|b|)$ holds for all $a, b \in A$.*

Proof. Let $(u_\alpha)_{\alpha\in\Lambda_{I^+}}$ be the approximate unit from lemma 1.5, and let $x_\alpha = u_\alpha a$. Note that $x_\alpha \in I$ since I is a two sided ideal and $u_\alpha \in I$ for all α. Also note that, $\|u_\alpha\|^2 \leq 1$. Thus,

$$|x_\alpha|^2 = x_\alpha^* x_\alpha = a^* u_\alpha^2 a \leq a^* \left\|u_\alpha^2\right\| a \leq |a|^2$$

by Murphy (2.2.5). Thus, since taking roots of positive elements preserves inequalities, $|x_\alpha| \leq |a|$ for all α, and so

$$\limsup_{\alpha \in \Lambda_{I+}} \tau(|x_\alpha|) \leq \tau(|a|). \tag{9}$$

Next, since $x_\alpha \xrightarrow{\|\cdot\|} a$ and $a \mapsto |a|$ is continuous by Rødam (1.2.5), it follows that $|x_\alpha| \xrightarrow{\|\cdot\|} |a|$. So by (9) and lower semicontinuity,

$$\tau(|a|) \leq \liminf_{\alpha \in \Lambda_{I+}} \tau(|x_\alpha|).$$

Thus, we have that

$$\lim_{\alpha \in \Lambda_{I+}} \tau(|x_\alpha|) = \tau(|a|). \tag{10}$$

Similarly, if we let $y_\alpha = u_\alpha b$, then $\lim_{\alpha \in \Lambda_{I+}} \tau(|y_\alpha|) = \tau(|b|)$. Observe that, $x_\alpha + y_\alpha = u_\alpha(a+b)$ so using the continuity of the map $a \mapsto |a|$ once more we have $|x_\alpha + y_\alpha| \xrightarrow{\|\cdot\|} |a+b|$. Hence, by lower semicontinuity and 1.6.6 we have that,

$$\tau(|a+b|) \leq \liminf_{\alpha \in \Lambda_{I+}} \tau(|x_\alpha + y_\alpha|) \leq \liminf_{\alpha \in \Lambda_{I+}} \Big(\tau(|x_\alpha|) + \tau(|y_\alpha|)\Big) = \tau(|a|) + \tau(|b|).$$

□

Claim 1.6.8. *For $a \in A$, if $|a| \in I$, then $|a^*| \in I$.*

Proof. Define w_n as before. Let $\epsilon > 0$, and let

$$g_n(x) = \left(\frac{1}{n} + x\right)^{-\frac{1}{2}} x^{\frac{1}{2}} \left(\frac{1}{n} + x\right)^{-\frac{1}{2}}$$

on $[0, \|a\|^2]$. Then by Stone-Weierstrass, for each n there exists a polynomial f_n such that $|g_n(x) - f_n(x)| < \frac{\epsilon}{2\|a\|^2}$ for all $x \in [0, \|a\|^2]$. Note that for any polynomial f we have that

$$af(a^*a)a^* = a\left(\sum_{k=1}^{\ell} a_k (a^*a)^k\right)a^* = \sum_{k=1}^{\ell} a_k a (a^*a)^k a^*$$

$$= \sum_{k=1}^{\ell} a_k (aa^*)^k aa^* = \left(\sum_{k=1}^{\ell} a_k (aa^*)^k \right) aa^* = f(aa^*)aa^*$$

Observe that,

$$\|ag_n(a^*a)a^* - g_n(aa^*)aa^*\|$$

$$= \|ag_n(a^*a)a^* - af(a^*a)a^* + f(aa^*)aa^* - g_n(aa^*)aa^*\|$$

$$\leq \|a\| \|g_n(a^*a) - f_n(a^*a)\| \|a\| + \|a\|^2 \|g_n(aa^*) - f_n(aa^*)\| < \epsilon.$$

Thus,

$$w_n |a| w_n^* = ag_n(a^*a)a^* = g_n(aa^*)aa^* = \left(\frac{1}{n} + aa^*\right)^{-1} (aa^*)^{\frac{3}{2}}$$

and so $w_n |a| w_n^* \xrightarrow{\|\cdot\|} |a^*|$ by the functional calculus. Hence, by lower semicontinuity,

$$\tau(|a^*|) \leq \liminf_{n\to\infty} \tau(w_n |a| w_n^*) \overset{1.6.3}{=} \liminf_{n\to\infty} \tau(|a| w_n^* w_n)$$

$$\overset{1.6.5}{\leq} \liminf_{n\to\infty} \tau(|a|) \|w_n^* w_n\| \leq \tau(|a|)$$

and so $|a^*| \in I$. □

Claim 1.6.9. *For $a \in A$, if $|a| \in I$, then $a \in I$.*

Proof. First, if $a \in A_{sa}$ then $|a| = a^+ + a^-$ where a^+ and a^- are defined in the proof of 1.6.1. Then, $|a| - a^+ = a^- \in A^+$ and $|a| - a^- = a^+ \in A^+$. Thus, since I is hereditary by 1.6.2, $a^+, a^- \in I$, and so $a^+ - a^- = a \in I$.

Next, for a general $a \in A$ we separate a into its real and imaginary parts, $a = x + iy$. Then,

$$2\tau(|x|) = \tau(|x + iy + x - iy|) \overset{1.6.7}{\leq} \tau(|a|) + \tau(|a^*|) < \infty,$$

by the previous claim. Thus, by the self-adjoint case, $x \in I$ and similarly for y and so $a = x + iy \in I$. □

Claim 1.6.10. *The map $|||\cdot||| : I \to [0, \infty)$ given by $|||a||| = \|a\| + \tau(|a|)$ is a sub-multiplicitive norm for I which makes I a non-unital Banach algebra. Moreover, this norm satisfies the inequality $|||xy||| \leq \|x\| \, |||y||| + |||x||| \, \|y\|$.*

Proof. First we show that $|||\cdot|||$ is a norm. Observe that:

1. $|||\lambda a||| = \|\lambda a\| + \tau(|\lambda a|) \leq |\lambda| \left(\|a\| + \tau(|a|) \right)$ since τ is linear.

2. Since $\|a\| \geq 0$ and $\tau(|a|) \geq 0$ (since $|a| \in I^+$), $|||a||| \geq 0$. Moreover, if $|||a||| = 0$ then $\|a\| = 0$ by positivity and so $a = 0$ since $\|\cdot\|$ is a norm. Next, $0 \in I$ whenever I is nonempty. Then since $|0| = (0^*0)^{\frac{1}{2}} = 0$, $\tau(|0|) = \tau(|0|) + \tau(|0|)$ implies $\tau(|0|) = 0$ so that $|||0||| = 0$.

3. $|||a+b||| = \|a+b\| + \tau(|a+b|) \overset{1.6.6}{\leq} \|a\| + \|b\| + \tau(|a|) + \tau(|b|) = |||a||| + |||b|||$

4. $|||ab||| = \|ab\| + \tau(|ab|) \leq \|a\|\,\|b\| + \tau(|ab|) \overset{(7)}{\leq} \|a\|\,\|b\| + \tau(|a|)\,\|b\|$
$= |||a|||\,\|b\| \leq |||a|||(\|b\| + \tau(|b|)) = |||a|||\,|||b|||$.

To show completeness suppose that $\{a_n\}$ is a Cauchy sequence under $|||\cdot|||$. Then given $\epsilon > 0$ there exists an $N > 0$ such that

$$|||a_n - a_m||| = \|a_n - a_m\| + \tau(|a_n - a_m|) < \epsilon$$

whenever $n, m \geq N$. This means that $\{a_n\}$ is Cauchy under $\|\cdot\|$ and so converges to some $a \in A$. Additionally, this implies that $\tau(|a_n - a_m|) < \epsilon$ whenever $m, n \geq N$. Moreover, 1.6.6 implies that $\tau(|a_n|) - \tau(|a_N|) < \tau(|a_n - a_N|)$. Putting this together we have that $\tau(|a_n|) < \tau(|a_N|) + \epsilon$ for all $n \geq N$. Thus, the sequence $\{\tau(|a_n|)\}$ is bounded so by lower semicontinuity $|a| \in I$. Hence, by 1.6.9, $a \in I$. Moreover, since $\tau(|a_N - a_n|) < \epsilon$ whenever $n \geq N$ lower semicontinuity once more gives $\tau(|a_n - a|) \to 0$. Thus, $a_n \to a$ under $|||\cdot|||$ and our space is complete.

Finally, $|||ab||| = \|ab\| + \tau(|ab|) \leq \|a\|\,\|b\| + \tau(|ab|) \overset{(7)}{\leq} \|a\|\,\|b\| + \tau(|a|)\,\|b\| = |||a|||\,\|b\| \leq |||a|||\,\|b\| + \|a\|\,|||b|||$ so the inequality is satisfied. □

Remark. Note that if A is unital then $I = A$ since the invertibles are open and I is a dense ideal. Thus, our unbounded trace is just a trace and so what we have developed is only interesting in the non-unital case.

2 K-theory and Dense Subalgebras

Now that we have a dense subalgebra I which is the domain of an unbounded trace τ we would like to know: Under what conditions a dense subalgebra will have the same K-theory as its parent algebra. Moreover, does our dense subalgebra I meet those conditions? For now we will assume that our parent algebra A is unital which, as we will see, will not make a difference when we switch to the K-theory in the non-unital case.

Lemma 2.1. *For any dense unital subalgebra B of a unital Banach algebra A the following are equivalent.*

i) B is inverse closed in A.

ii) Every maximal right ideal $\mathfrak{n}$ of B is a relatively closed subset. That is, $\mathfrak{n} = B \cap \overline{\mathfrak{n}}$.

iii) Every irreducible B-module N extends to an A-module M; that is, N is a B-submodule of the restriction of M to B.

Proof. ***i) $\Longrightarrow$ ii):*** Since B is inverse closed in A, $\mathfrak{n} \cap \operatorname{Inv}(A) = \emptyset$ for every maximal right ideal $\mathfrak{n}$ of B. By definition, every neighborhood of $a \in \overline{\mathfrak{n}}$ contains a point of $\mathfrak{n}$. Since $\operatorname{Inv}(A)$ is open, $\overline{\mathfrak{n}} \cap \operatorname{Inv}(A) = \emptyset$. Next, we show that $\overline{\mathfrak{n}}$ is a right ideal in A. Since B is dense in A and addition and multiplication are continuous, given any $\alpha a + \beta b$ where $\alpha, \beta \in \overline{\mathfrak{n}}$ we may construct a sequence

$$\alpha_n a_n + \beta_n b_n \text{ such that } \|\alpha_n a_n + \beta_n b_n - (\alpha a + \beta b)\| < 2^{-n}$$

where $\alpha_n, \beta_n \in \mathfrak{n}$ and $a_n, b_n \in B$. Thus, $\alpha a + \beta b \in \overline{\mathfrak{n}}$ so that $\overline{\mathfrak{n}}$ is an ideal in A. Then it follows that $\overline{\mathfrak{n}} \cap B$ is an ideal in B. Moreover, since $\overline{\mathfrak{n}} \cap B$ does not contain the unit it is a proper ideal of B. Thus, by maximality, $\overline{\mathfrak{n}} \cap B = \mathfrak{n}$ and so $\mathfrak{n}$ is relatively closed.

ii) $\Longrightarrow$ iii): Suppose that N is an irreducible B-module. Define $\varphi_n : B \to N$ by $a \mapsto na$ where $n \in N$. Then for any nonzero $n \in N$ we have that $\varphi_n(1) = n \neq 0$ so that φ_n is nonzero whenever $n \neq 0$. Fix a nonzero $n \in N$. Since the image of φ_n is a submodule of the irreducible module N, we must have that φ_n is surjective. Let $\mathfrak{n}$ be the kernel of φ_n. Then ψ_n, the map induced by φ_n on $\mathfrak{n}\backslash B$, is an isomorphism. Hence, $\mathfrak{n}$ is a maximal right ideal of B.

Next, let $\mathfrak{m} = \overline{\mathfrak{n}}$. Then by *ii)*, $\mathfrak{n} = \mathfrak{m} \cap B$. So, by the second isomorphism theorem,

$$N \cong {}_{\mathfrak{m}\cap B}\backslash B \cong {}_{\mathfrak{m}}\backslash \mathfrak{m} + B \subseteq {}_{\mathfrak{m}}\backslash A = M.$$

iii) $\Longrightarrow$ i): We show the contrapositive. Suppose that B is <u>not</u> inverse closed in A. That is, there exists an $a \in B$ that is invertible in A but not invertible in B. Since a is invertible in A, a cannot have a right inverse in B by the uniqueness of inverses. Consider the right ideal generated by a, $(a) = aB$. Since a is not right invertible $\mathbb{1} \notin (a)$ so that (a) is a proper right ideal. Thus,

(a) is contained in a maximal right ideal, say $\mathfrak{n} \in B$. Then ${}_{\mathfrak{n}}\diagdown^{B}$ is an irreducible B-module. Letting $[\mathbb{1}] \in {}_{\mathfrak{n}}\diagdown^{B}$ be the class of the identity of B we have that,

$$[\mathbb{1}]a = a = 0, \text{ i.e. } (\mathfrak{n} + \mathbb{1})a = \mathfrak{n} + a = \mathfrak{n} \text{ since } a \in \mathfrak{n} \tag{11}$$

Note that for any A-module M, since $a \in \mathrm{Inv}(A)$ the map $M \to M$ defined by $m \mapsto ma$ is injective. Thus, if ${}_{\mathfrak{n}}\diagdown^{B}$ were contained in the restriction of M to B, by (11) we would have that $[\mathbb{1}] \mapsto 0$. Since this map is injective that would mean $[\mathbb{1}] = 0$, a contradiction. □

Corollary 2.1.1. *Let B be a dense unital subalgebra of the unital Banach algebra A (with the same unit). Then $M_n(B)$ is inverse closed in $M_n(A)$ whenever B is inverse closed in A for all n.*

Proof. Let $\mathbf{e}_{ij}$ be the $n \times n$ matrix with 1_A in the ij'th entry and zeros elsewhere and let id_n be the identity matrix in $M_n(B)$. Note that for any $M_n(B)$-module, say W, $W = id_n(W) = \sum_{i=1}^{n} \mathbf{e}_{ii}(W)$. Moreover, if $x \in \mathbf{e}_{ii}(W) \cap \mathbf{e}_{jj}(W)$ where $i \neq j$, then $\mathbf{e}_{ii}y = x = \mathbf{e}_{jj}y'$ for some $y, y' \in W$. Thus, $0 = \mathbf{e}_{ii}\mathbf{e}_{jj}y' = \mathbf{e}_{ii}\mathbf{e}_{ii}y = \mathbf{e}_{ii}y = x$. So $\mathbf{e}_{ii}(W) \cap \mathbf{e}_{jj}(W) = \{0\}$ whenever $i \neq j$. Furthermore, $\mathbf{e}_{ii}(W) \cong \mathbf{e}_{jj}(W) \cong N$ for some B-module N by means of the shift operator. Hence, any $M_n(B)$-module W is the direct sum $N^{\bigoplus n}$ for some B-module N. Next, note that any submodule $V \leq W$ is of the form $L^{\bigoplus n}$ for some B-module L which is a submodule of N. Thus, $V \leq W$ if and only if $L \leq N$ and so; any $M_n(B)$-module $W = N^{\bigoplus n}$ is irreducible if and only if N is an irreducible B-module.

By supposition B is inverse closed in A, so by the previous lemma every irreducible B-module N extends to a A-module M. Since every irreducible $M_n(B)$-module is of the form $N^{\bigoplus n}$ where N is a irreducible B-module it follows that $N^{\bigoplus n}$ extends to the $M_n(A)$-module $M^{\bigoplus n}$. Thus, by the previous lemma once more, $M_n(B)$ is inverse closed in $M_n(A)$. □

Theorem 2.2. *Let A be a unital C^*-algebra and let $B \subseteq A$ be a dense unital Banach $*$-subalgebra such that*

i) There is a topology on B stronger then the topology B inherits from A under which B is a Banach algebra.

ii) B is inverse closed in A.

Then the natural map induced by inclusion, $\iota_* : K_0(B) \to K_0(A)$, is an isomorphism.

proof of surjectivity. First we show that the map is surjective. Note that it suffices to show that for an element of the form $[p] \in K_0(A)$ there exists a $[q] \in K_0(B)$ such that $[p] = [q]$ since such elements generate $K_0(A)$ and $K_0(B)$. Thus, for $[p] \in K_0(A)$, $p = (a_{ij})$ is a projection in $M_n(A)$ for some n. Since B is dense in A, given $\epsilon > 0$ there exists a b_{ij} for each a_{ij} such that $\|a_{ij} - b_{ij}\| < \frac{\epsilon}{n^2}$. Let $y = (b_{ij}) \in M_n(B)$. Then by the triangle inequality, $\|p - y\| \leq \sum_{i,j=1}^{n} \|a_{ij} - b_{ij}\| < \epsilon$. Take $x = \frac{1}{2}(y + y*)$, then

$$\|x - p\| = \left\| \frac{1}{2}(y - p + y^* - p^*) \right\| \leq \frac{1}{2}\|y - p\| + \frac{1}{2}\|(y - p)^*\| \leq \epsilon.$$

Thus, for any $\epsilon > 0$ there exists a self-adjoint $x \in M_n(B)$ such that $\|x - p\| < \epsilon$. Note that $\|x\| \leq \epsilon + \|p\|$; and so,

$$\left\|x^2 - x\right\| = \left\|x^2 - p + p - x\right\| \leq \left\|x^2 - p^2\right\| + \|p - x\| = \|(x + p)(x - p)\| + \|x - p\|$$

$$< \epsilon\|x + p\| + \epsilon \leq \epsilon(\epsilon + \|p\| + \|p\|) + \epsilon = \epsilon^2 + \epsilon(2\|p\| + 1) < \epsilon^2 + 3\epsilon.$$

Hence, we may take ϵ small enough so that $\left\|x^2 - x\right\| < \eta$ where $\eta \in (0, \frac{1}{4}]$. Thus, by the functional calculus, we may take η small enough that $\sigma(x) \subseteq (-\frac{1}{n}, \frac{1}{n}) \cup (1 - \frac{1}{n}, 1 + \frac{1}{n})$, $n \geq 4$. Let q be the spectral projection of x corresponding to the interval $(1 - \frac{1}{n}, 1 + \frac{1}{n}) = K_n$; i.e. $\chi_{K_n}(x)$. Then q is a projection since

$$\chi_{K_n}(t) = (\chi_{K_n}(t))^2 = \overline{\chi_{K_n}(t)} \text{ for any } t \in \sigma(x).$$

Moreover, $|t - \chi_{K_n}(t)| < \frac{1}{n}$ for all $t \in \sigma(x)$ so that $\|x - q\| < \frac{1}{n}$. Then $\|q - p\| \leq \|q - x\| + \|x - p\| < \frac{1}{n} + \epsilon$. Taking $\epsilon \leq \frac{1}{n}$ we have that $\|p - q\| < \frac{2}{n}$. Take $n = 4$. Note that if p and q are conjugate (possibly after stabilization) they are both representatives of the same class in $K_0(A)$. Let $u = pq + (1 - p)(1 - q)$. Then $pu = pq = uq$ so to show that $[p] = [q]$ we need only show that u is invertible. Observe that, $u = pq + 1 - p - q + pq = 1 + p - q - 2p + 2pq = 1 + (1 - 2p)(p - q)$, and since p is a projection the functional calculus shows that $\|1 - 2p\| = 1$. Hence,

$$\|1 - u\| \leq \|1 - 2p\| \|p - q\| < \frac{1}{2}$$

so that u is invertible and $[p] = [q]$ in $K_0(A)$.

It remains to be shown that $q \in M_n(B)$. Let Γ be a smooth closed convex curve in the resolvent set of x encircling $(\frac{1}{2}, \frac{3}{2})$ but not $(-\frac{1}{2}, \frac{1}{2})$ parameterized by t on $[a, b]$. For instance, the circle centered at 1 of radius 1/2 given by $z(t) =$

$1+\frac{1}{2}e^{it}$ for $t \in [0, 2\pi]$. Form a partition of $[a, b]$, i.e. $a = t_0 < t_1 < \cdots < t_n = b$, where $t_{j-1} \leq t'_j \leq t_j$. Consider the limit of the Riemann sum

$$\frac{1}{2\pi i}\sum_{j=1}^{n}\Big(\mathbb{1}z(t'_j) - x\Big)^{-1}[z(t_j) - z(t_{j-1})] \tag{12}$$

as $|P| := \max\{t_j - t_{j-1} : j = 1, \cdots, n\} \to 0$. Note that this sum converges in the norm of B to the line integral of the Banach space valued function $(\mathbb{1}z(t) - x)^{-1}$, i.e. $\frac{1}{2\pi i}\int_\Gamma(\mathbb{1}z(t) - x)^{-1}dz$. Indeed, Since since $\mathbb{1}, x \in B$ and B is inverse closed, we have that $(\mathbb{1}z(t) - x)^{-1} \in B$. Then since inversion is continuous on the invertible elements of B (Murphy 1.2.3) and $[a, b]$ is compact, $t \mapsto (\mathbb{1}z(t) - x)^{-1}$ is uniformly continuous. Hence, there exists a δ such that $\left\|(\mathbb{1}z(t_j) - x)^{-1} - (\mathbb{1}z(t_i) - x)^{-1}\right\|_B < \frac{\epsilon}{|\Gamma|}$ whenever $|t_j - t_i| < \delta$ Thus, given any two partitions P and Q such that $|P|, |Q| < \delta$ we have that,

$$\left\|\sum_{j=1}^{n}\Big(z(t'_j) - x\Big)^{-1}[z(t_j) - z(t_{j-1})] - \sum_{i=1}^{m}\Big(z(t'_i) - x\Big)^{-1}[z(t_i) - z(t_{i-1})]\right\|_B$$

$$< \frac{\epsilon}{|\Gamma|}\sum_{k=1}^{\ell}|z(t_k) - z(t_{k-1})| \leq \epsilon$$

where $\{t_k\}$ is the refinement of the partitions P, Q.

Next, let φ be any multiplicative linear functional on the C^*-algebra generated by x and $\mathbb{1}$. Since $\chi_K(x)$ is the limit of polynomials in x and $\mathbb{1}$, we see that $\varphi(\chi_K(x)) = \chi_K(\varphi(x))$. Also, by considering (12) it follows that,

$$\varphi\Big(\frac{1}{2\pi i}\int_\Gamma(\mathbb{1}z(t) - x)^{-1}dz\Big) = \frac{1}{2\pi i}\int_\Gamma(z(t) - \varphi(x))^{-1}dz.$$

Hence,

$$\varphi(\chi_K(x)) = \chi_K(\varphi(x)) = \frac{1}{2\pi i}\int_\Gamma(z(t) - \varphi(x))^{-1}dz = \varphi\Big(\frac{1}{2\pi i}\int_\Gamma(\mathbb{1}z(t) - x)^{-1}dz\Big)$$

Thus, since multiplicative linear functionals separate points,

$$q = \chi_K(x) = \frac{1}{2\pi i}\int_\Gamma(z(t) - x)^{-1}dz$$

so that $q \in M_n(B)$ and our map is surjective. □

proof of injectivity. Suppose that $[p_1], [p_2]$ are in $K_0(B)$ such that $[p_1] = [p_2]$ in $K_0(A)$. We need to show that $[p_1] = [p_2]$ in $K_0(B)$. Since $[p_1] = [p_2]$ in $K_0(A)$ there exists a projection $r \in M_m(A)$ and a $n \in \mathbb{N}$ such that (possibly by passing to a larger matrix by adding zeros) $(p_1 \oplus r)$ is conjugate to $(p_2 \oplus r)$ in $M_n(A)$. That is, there is a $u \in GL_n(A)$ such that $u(p_1 \oplus r)u^{-1} = (p_2 \oplus r)$. Let $\epsilon > 0$ be given. Note that we may construct a projection $r_0 \in M_m(B)$ as we did for q in the proof of surjectivity such that

$$\|r - r_0\| < \min\Big\{\frac{\epsilon}{4}, \frac{\epsilon}{4\,\|u\|\,\|u^{-1}\|}\Big\}.$$

Let

$$\gamma_1 = \frac{\epsilon}{4\,\|p_1 \oplus r_0\|\,\|u^{-1}\|},\ \gamma_2 = \frac{\epsilon\,\|u^{-1}\|}{4\,\|u\|\,\|u^{-1}\|\,\|p_1 \oplus r_0\| + \epsilon} = \frac{\epsilon}{4(\|u\| + \gamma_1)\,\|p_1 \oplus r_0\|}.$$

Next, by the continuity of inversion, there exists a $\delta > 0$ such that $\left\|u^{-1} - v^{-1}\right\| < \gamma_2$ whenever $\|u - v\| < \delta$. Since $GL_n(A)$ is open in $M_n(A)$, by the density of B in A we can choose a $v \in GL_n(B)$ such that $\|u - v\| < \min\{\delta, \gamma_1\}$. Then since $\|v\| < \|u\| + \gamma_1$ we have that,

$$\begin{aligned}
&\left\|v(p_1 \oplus r_0)v^{-1} - (p_2 \oplus r_0)\right\| \\
&\le \left\|v(p_1 \oplus r_0)(v^{-1} - u^{-1})\right\| + \left\|(v - u)(p_1 \oplus r_0)u^{-1}\right\| \\
&+ \left\|u\Big((p_1 \oplus r_0) - (p_1 \oplus r)\Big)u^{-1}\right\| + \|(p_2 \oplus r) - (p_2 \oplus r_0)\| \\
&< (\|u\| + \gamma_1)\,\|p_1 \oplus r_0\|\,\left\|v^{-1} - u^{-1}\right\| + \|v - u\|\,\|p_1 \oplus r_0\|\,\left\|u^{-1}\right\| \\
&+ \|u\|\,\left\|u^{-1}\right\|\,\|r_0 - r\| + \|r_0 - r\| < \epsilon.
\end{aligned}$$

Next, let $q = v(p_1 \oplus r_0)v^{-1},\ p = p_2 \oplus r_0$, and let $w = pq + (1 - p)(1 - q)$. By the same argument as in the surjective case, $pw = wq$ and w is invertible whenever $\epsilon < 1$. Note that $w, w^{-1} \in M_n(B)$ since $q, p \in M_n(B)$. Thus, since $p_2 \oplus r_0 = (wv)(p_1 \oplus r_0)(wv)^{-1}$ we have that $[p_1 \oplus r_0] = [p_2 \oplus r_0] \implies [p_1] = [p_2]$ in $K_0(B)$ and so our map is injective. $\square$

Corollary 2.2.1. *Let A be a unital C^*-algebra and let $B \subseteq A$ be a dense unital $*$-subalgebra which becomes a Banach algebra in some norm $\|\cdot\|_B \ge \|\cdot\|_A$. Suppose that there is some constant $C \ge 1$ such that*

$$\|ab\|_B \le C\Big(\,\|a\|_A\,\|b\|_B + \|a\|_B\,\|b\|_A\Big) \quad \text{for all} \quad a, b \in B. \tag{13}$$

Then the natural map induced by inclusion $K_0(B) \to K_0(A)$, is an isomorphism.

Proof. By the previous theorem we need only show inverse closure. For an arbitrary $a \in B$, applying (13) to $x = y = a^n$ we have $\left\|a^{2n}\right\|_B \leq 2C \left\|a^n\right\|_B \left\|a^n\right\|_A$. Note that if $p(x) = x^2$ then since $p(\sigma(a)) = \sigma(p(a))$, if $\lambda \in \sigma(a)$ then $\lambda^2 \in \sigma(p(a))$. Thus, if $r_B(a)$ is the spectral radius of $a \in B$, then $r_B(a^2) = (r_B(a))^2$. Hence,

$$(r_B(a))^2 = \lim_{n\to\infty} \left\|a^{2n}\right\|_B^{\frac{1}{n}} \leq \lim_{n\to\infty} (2C)^{\frac{1}{n}} \left\|a^n\right\|_B^{\frac{1}{n}} \left\|a^n\right\|_A^{\frac{1}{n}} = r_B(a) r_A(a),$$

so that $r_B(a) \leq r_A(a)$ by the spectral radius formula. Next, if an element $c \in B$ is not invertible in A then it is not invertible in B so that $\sigma_A(a) \subseteq \sigma_B(a)$ and so $r_A(a) \leq r_B(a)$. Therefore, $r_B(a) = r_A(a)$.

Now if a is invertible in A then a^*a is invertible in A. Since a^*a is positive and invertible, $\sigma_A(a^*a) \subseteq [\eta, \|a\|^2]$ for some $\eta \in \mathbb{R}_{>0}$. Let $\lambda > \|a\|^2$. Observe that,

$$\mu \in \sigma_A(\lambda - a^*a) \iff \mu - \lambda \in \sigma_A(-a^*a) \iff \lambda - \mu \in \sigma_A(a^*a) \subseteq [\eta, \|a\|^2].$$

It follows that, $\|a\|^2 \geq \lambda - \mu > \|a\|^2 - \mu$, so $\mu > 0$. Moreover, $\lambda - \mu \geq \eta > 0$ so that for all $\mu \in \sigma_A(\lambda - a^*a)$, $\lambda > \mu > 0$. Hence, $r_B(\lambda - a^*a) = r_A(\lambda - a^*a) < \lambda$, that is $\lambda \notin \sigma_B(\lambda - a^*a)$ and so $a^*a = \lambda - (\lambda - a^*a)$ is invertible in B. Suppose c is the inverse of a^*a in B. Then ca^* is a left inverse of a in B. Since ca^* is a left inverse of a in A also, it must be the inverse in A and so a is invertible in B. Thus, B is inverse closed. □

Now we return to the case where A and B are non-unital.

Lemma 2.3. *Suppose that B is a nonunital Banach $*$-algebra and A is a nonunital C^*-algebra containing B. If B is dense in A then $\widetilde{B}$ is dense in $\widetilde{A}$. Moreover, if $\|\cdot\|_B \geq \|\cdot\|_A$ then $\|\cdot\|_{\widetilde{B}} \geq \|\cdot\|_{\widetilde{A}}$.*

Proof. Let $\epsilon > 0$ and (a, λ) be given. Since B is dense in A, given any $a \in A$ there exists a $b \in B$ such that $\|a - b\|_A < \epsilon$. Thus, $\|(a, \lambda) - (b, \lambda)\|_{\widetilde{A}} = \|(a, 0) - (b, 0)\|_{\widetilde{A}} = \|a - b\|_A < \epsilon$, since the inclusion $a \mapsto (a, 0)$ is an isometric $*$-homomorphism.

Next, recall that if B is a Banach algebra, setting $\|(b, \lambda)\|_{\widetilde{B}} = \|b\|_B + |\lambda|$ makes $\widetilde{B}$ a Banach algebra. Observe that, $\|(b, \lambda)\|_{\widetilde{A}} \leq \|(b, 0)\|_{\widetilde{A}} + \|(0, \lambda)\|_{\widetilde{A}} = \|b\|_A + |\lambda| \leq \|b\|_B + |\lambda| = \|(b, \lambda)\|_{\widetilde{B}}$. □

Lemma 2.4. *Suppose that B is a nonunital Banach $*$-algebra such that B is dense in A, a nonunital C^*-algebra. Suppose also that*

$$\|ab\|_B \le \|a\|_A \|b\|_B + \|a\|_B \|b\|_A \quad \text{for all} \quad a, b \in B. \tag{14}$$

Then for $\lambda, \mu \in \mathbb{C}$ we have,

$$\|(a,\lambda)(b,\mu)\|_{\widetilde{B}} \le 6\Big(\|(a,\lambda)\|_{\widetilde{A}} \|(b,\mu)\|_{\widetilde{B}} + \|(a,\lambda)\|_{\widetilde{B}} \|(b,\mu)\|_{\widetilde{A}} \Big)$$

Proof. First, let $x \in A_{sa}$ and $r \in \mathbb{R}$. Define $f_r(y) = y + r$, which is continuous on $\sigma_{\widetilde{A}}((x,0))$. Moreover, $f_r((x,0)) = (x,r)$, and so

$$\sigma_{\widetilde{A}}((x,r)) = \sigma_{\widetilde{A}}(f_r((x,0))) = f_r(\sigma_{\widetilde{A}}((x,0))) = \sigma_A(x) + r.$$

Let $b = \sup \sigma_A(x)$, and let $a = \inf \sigma_A(x)$. Then since $x \in A_{sa}$ and A is nonunital, $\|x\|_A = -a$ or b. Moreover, since $(x,r) \in \widetilde{A}_{sa}$,

$$\|(x,r)\|_{\widetilde{A}} = \max\{\left|\sup \sigma_{\widetilde{A}}((x,r))\right|, \left|\inf \sigma_{\widetilde{A}}((x,r))\right|\}$$

$$= \max\{|\sup \sigma_A(x) + r|, |\inf \sigma_A(x) + r|\}$$

Observe that,

$$\left\|\left(x, \frac{a-b}{2} - a\right)\right\|_{\widetilde{A}} = \left|\sup \sigma_A(x) + \frac{a-b}{2} - a\right| = \left|\inf \sigma_A(x) + \frac{a-b}{2} - a\right| = \frac{b-a}{2}$$

Next, if $r > \frac{a-b}{2} - a$ then $b + r > \frac{b-a}{2} \ge 0$, so that

$$\left\|\left(x, \frac{a-b}{2} - a\right)\right\|_{\widetilde{A}} = \frac{b-a}{2} < b + r \le \|(x,r)\|_{\widetilde{A}}.$$

On the other hand, if $r < \frac{a-b}{2} - a$ then $a + r < \frac{a-b}{2} \le 0$, so that

$$\left\|\left(x, \frac{a-b}{2} - a\right)\right\|_{\widetilde{A}} = \frac{b-a}{2} < |a + r| \le \|(x,r)\|_{\widetilde{A}}.$$

Note that,

$$\frac{\|x\|_A - a}{2} \ge \frac{1}{2}\|x\|_A \text{ and that } \frac{b + \|x\|_A}{2} \ge \frac{1}{2}\|x\|_A.$$

Thus, since $\|x\|_A = -a$ or b, we have that

$$\|x\|_A \leq b - a = 2 \left\| \left(x, \frac{a-b}{2} - a \right) \right\|_{\widetilde{A}} \leq 2 \|(x, r)\|_{\widetilde{A}}$$

for all $x \in A_{sa}$ and all $r \in \mathbb{R}$.

Next, for a general element $a \in A$ we write $a = x + iy$ Then for $\lambda \in \mathbb{C}$ by the self adjoint case we have,

$$\|a\|_A \leq \|x\|_A + \|y\|_A \leq 2 \|(x, \mathrm{Re}(\lambda))\|_{\widetilde{A}} + 2 \|(y, \mathrm{Im}(\lambda))\|_{\widetilde{A}}.$$

Since

$$\|(x, \mathrm{Re}(\lambda))\|_{\widetilde{A}} = \frac{1}{2} \|(a, \lambda) + (a, \lambda)^*\|_{\widetilde{A}} \leq \|(a, \lambda)\|_{\widetilde{A}}$$

and similarly, $\|(y, \mathrm{Im}(\lambda))\|_{\widetilde{A}} \leq \|(a, \lambda)\|_{\widetilde{A}}$, we have that,

$$\|a\|_A \leq 4 \|(a, \lambda)\|_{\widetilde{A}} \text{ for all } a \in A \text{ and all } \lambda \in \mathbb{C} \tag{15}$$

Note that $\lambda \in \sigma(a, \lambda)$ since $(0, \lambda) - (a, \lambda) = (a, 0)$ is not invertible. Thus, for all $\lambda \in \mathbb{C}$ and all $a \in A$ we have that,

$$|\lambda| \leq \|(a, \lambda)\|_{\widetilde{A}} \tag{16}$$

Putting all this together we have,

$$\|(a, \lambda)(b, \mu)\|_{\widetilde{B}} \leq \|ab + \lambda b + a\mu\|_B + |\lambda\mu| \leq \|ab\| + |\lambda| \|b\|_B + |\mu| \|a\|_B + |\lambda\mu|$$

$$\overset{(14)}{\leq} \|a\|_A \|b\|_B + \|a\|_B \|b\|_A + |\lambda| \|b\|_B + |\mu| \|a\|_B + |\lambda\mu|$$

$$= \|a\|_B \Big(\|b\|_A + |\mu| \Big) + \|b\|_B \Big(\|a\|_A + |\lambda| \Big) + |\lambda\mu|$$

$$\leq \Big(\|a\|_B + |\lambda| \Big) \Big(\|b\|_A + |\mu| \Big) + \Big(\|b\|_B + |\mu| \Big) \Big(\|a\|_A + |\lambda| \Big) + |\lambda\mu|$$

$$\overset{(15)}{\leq} \|(a, \lambda)\|_{\widetilde{B}} \Big(4 \|(b, \mu)\|_{\widetilde{A}} + |\mu| \Big) + \|(b, \mu)\|_{\widetilde{B}} \Big(4 \|(a, \lambda)\|_{\widetilde{A}} + |\lambda| \Big) + |\lambda\mu|$$

$$\leq 4 \Big(\|(a, \lambda)\|_{\widetilde{B}} \|(b, \mu)\|_{\widetilde{A}} + \|(b, \mu)\|_{\widetilde{B}} \|(a, \lambda)\|_{\widetilde{A}} \Big) + |\mu| \|(a, \lambda)\|_{\widetilde{B}} + |\lambda| \|(b, \mu)\|_{\widetilde{B}} + |\lambda| |\mu|$$

$$\overset{(16)}{\leq} 5 \Big(\|(a, \lambda)\|_{\widetilde{B}} \|(b, \mu)\|_{\widetilde{A}} + \|(b, \mu)\|_{\widetilde{B}} \|(a, \lambda)\|_{\widetilde{A}} \Big) + \|(a, \lambda)\|_{\widetilde{A}} \Big(\|b\|_B + |\mu| \Big)$$

$$= 6 \Big(\|(a, \lambda)\|_{\widetilde{B}} \|(b, \mu)\|_{\widetilde{A}} + \|(b, \mu)\|_{\widetilde{B}} \|(a, \lambda)\|_{\widetilde{A}} \Big),$$

as was to be shown. $\square$

Theorem 2.5. *Let I be the ideal obtained from a unbounded trace τ from the previous section. Then the natural map induced by inclusion, $\iota_* : K_0(I) \to K_0(A)$, is an isomorphism.*

Proof. Note that the inclusion maps $\iota : I \hookrightarrow A$ and $\widetilde{\iota} : \widetilde{I} \hookrightarrow \widetilde{A}$ are algebra homomorphisms. Thus, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(I) & \longrightarrow & K_0(\widetilde{I}) & \longrightarrow & K_0(\mathbb{C}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \widetilde{\iota_*}} & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(\widetilde{A}) & \longrightarrow & K_0(\mathbb{C}) & \longrightarrow & 0 \end{array}$$

Since $|||\cdot|||_B$ satisfies the conditions of lemma 2.4, $|||\cdot|||_{\widetilde{B}}$ satisfies the conditions of corollary 2.2.1. Hence, we have shown that $\widetilde{\iota_*}$ is an isomorphism. Thus, by the short 5's lemma, ι_* is an isomorphism. $\square$

Theorem 2.6. *An unbounded trace on a C^*-algebra A induces a dimension function on $K_0(A)$.*

Proof. By the previous theorem ι_*^{-1} is an isomorphism and by claim 1.6.3 τ is a trace on I. Thus, by theorem 0.2 $(\dim_\tau \circ \iota_*^{-1})$ is an algebra homomorphism.

$\square$